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The dyon–electron system: scattering and electron capture

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Abstract. Two aspects of the non-relativistic dyon-electron system are studied, first the dyon-electron scattering and second, electron capture by a dyon.

1. Introduction

We continue the programme of studying various aspects of the non-relativistic dyon-electron system. In an earlier publication (Bose 1986) the bound-state problem was solved; the energy levels obtained and the bound-state wavefunctions constructed explicitly. Moreover, the light absorption characteristic of the system was considered and the absorption cross section computed for the case where the initial system is in the ground state. In this paper, we concentrate on problems associated with the non-bound dyon-electron system. Specifically, we consider the problems of scattering and electron capture by dyons. This paper is organised as follows. In the next section we obtain the positive-energy solutions of the eigenvalue equation. In section 3, the wavefunctions appropriate to the picture of an incoming fermion with a definite helicity are constructed. These results are utilised in the next two sections. In section 4 the scattering problem is solved. The resulting scattering amplitude is checked to possess correct limiting forms for two cases; namely, the monopole-fermion scattering and the Rutherford scattering. Finally, in section 5, electron capture by a dyon leading to a bound system in its ground state is considered. The capture cross section is obtained for the case where the magnetic charge of the dyon is one Dirac unit and the initial particle is slowly moving. The result is compared with the corresponding case of electron capture by a proton leading to hydrogen atom formation.

2. Positive-energy wavefunctions

We denote by M the electron mass and by $-e$ the electron charge. The dyon charge and magnetic pole strength are Q and g, respectively. We use natural units and set $q=-eg$. The Schrödinger–Pauli Hamiltonian

$$H=\frac{1}{2M}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2-\frac{eQ}{r} \tag{1}$$

with

$$\boldsymbol{\pi}=\boldsymbol{p}+e\boldsymbol{A} \tag{2}$$

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where A is the vector potential of the dyon (Dirac expression), leads to the eigenvalue equation

$$H\Psi = E\Psi \tag{3}$$

which is subjected to angular momentum analysis using the two-component angular eigensection $\xi^{(1)}_{jm}$, $\xi^{(2)}_{jm}$ and η_m of Kazama *et al* (1977). Equation (3) is then found to possess three types of positive energy ($E > 0$) solutions. Setting

$$k = (2ME)^{1/2} \qquad \lambda = -\frac{MeQ}{k} \qquad a = [(J+\tfrac{1}{2})^2 - q^2]^{1/2} \tag{4}$$

these solutions can be written as follows:

type 1

$$\Psi^{(1)}_{jm} = \mathrm{e}^{-\mathrm{i}kr} r^{a-1} {}_1F_1(a - \mathrm{i}\lambda, 2a, 2\mathrm{i}kr)\xi^{(1)}_{jm} \qquad J \geqslant |q| + \tfrac{1}{2} \tag{4a}$$

type 2

$$\Psi^{(2)}_{jm} = \mathrm{e}^{-\mathrm{i}kr} r^{a} {}_1F_1(a + 1 - \mathrm{i}\lambda, 2a+2, 2\mathrm{i}kr)\xi^{(2)}_{jm} \qquad J \geqslant |q| + \tfrac{1}{2} \tag{4b}$$

type 3

$$\Psi^{(3)}_{jm} = \mathrm{e}^{-\mathrm{i}kr} {}_1F_1(1 - \mathrm{i}\lambda, 2, 2\mathrm{i}kr)\eta_m \qquad J = |q| - \tfrac{1}{2}. \tag{4c}$$

In the above, ${}_1F_1$ is the confluent hypergeometric function.

We shall later need the asymptotic expressions of the radial wavefunction as $r \to \infty$. Here we note the general result:

$$|F|(\alpha, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \mathrm{e}^{z} z^{\alpha-\gamma}\left(1 + \frac{(\gamma-\alpha)(1-\alpha)}{z}\right) + \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha}\left(1 - \frac{\alpha(\alpha-\gamma+1)}{z}\right) + \mathrm{O}\left(\frac{1}{z^2}\right) \tag{5}$$

which is valid as $|z| \to \infty$.

3. Scattering wavefunctions

We consider an electron coming in from infinity along the positive z axis with the dyon fixed at the origin. The wavefunction describing the scattering of the helicity ± 1 electron will be constructed by a superposition of the wavefunction (4), obtained above.

$$^{\pm}\Psi = \sum_{J,m} {}^{\pm}K^{(1)}_{jm}\Psi^{(1)}_{jm} + \sum_{J,m} {}^{\pm}K^{(2)}_{jm}\Psi^{(2)}_{jm} + \sum_{m} {}^{\pm}K^{(3)}_{m}\Psi^{(3)}_{m}. \tag{6}$$

The coefficients $^{\pm}K$ will be fixed by the requirement that as $r \to \infty$, the incoming part of the right-hand side of (6) matches the incoming part of the incident 'Coulomb distorted plane wave'. The latter is easily constructed from the exact Coulomb wavefunction, and is found to be given by the expression

$$-\frac{\exp[-\mathrm{i}(kr - \lambda \ln 2kr)]}{\mathrm{i}kr}\,\delta(1-\cos\theta)\begin{pmatrix}0\\1\end{pmatrix} \tag{7a}$$

for helicity 1 and by

$$-\frac{\exp[-\mathrm{i}(kr - \lambda \ln 2kr)]}{\mathrm{i}kr}\,\delta(1-\cos\theta)\begin{pmatrix}1\\0\end{pmatrix} \tag{7b}$$

for helicity -1, as $r \to \infty$. Using now the expressions for the two-component spinors in terms of the complete set of angular sections given by Kazama *et al* (1977), we obtain the desired expressions, as $r \to \infty$, for the incoming wave:

$$[{}^{+}\Psi]_{\mathrm{inc}} \to -\frac{\exp[-\mathrm{i}(kr-\lambda\ln 2kr)]}{\mathrm{i}kr}\,\delta(1-\cos\theta)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= -\frac{\exp[-\mathrm{i}(kr-\lambda\ln 2kr)]}{\mathrm{i}kr}\sqrt{\pi}\left((|q|-q)^{1/2}\eta_m + \sum \frac{q\sqrt{2J+1}}{\sqrt{2}|q|}(\xi_{jm}^{(1)}+\xi_{jm}^{(2)})\right) \tag{8a}$$

where m stands for $-q-\frac{1}{2}$ and the summation extends from $J=|q|+\frac{1}{2}$ to ∞. For the negative helicity, we obtain

$$[{}^{-}\Psi]_{\mathrm{inc}} \to -\frac{\exp[-\mathrm{i}(kr-\lambda\ln 2kr)]}{\mathrm{i}kr}\sqrt{\pi}\left(-(|q|+q)^{1/2}\eta_m + \sum \frac{q(2J+1)^{1/2}}{\sqrt{2}|q|}(\xi_{jm}^{(1)}-\xi_{jm}^{(2)})\right) \tag{8b}$$

where m stands for $-q+\frac{1}{2}$ and the summation extends from $J=|q|+\frac{1}{2}$ to ∞. The above expressions are valid in the region a, in the notation of Kazama *et al* (1977) which is the sphere minus a strip containing the south pole. It is enough to match the wave sections in this region. We are now in a position to compute the coefficients ${}^{\pm}K$ that appear in (6). From the explicit expressions, given by equation (4), for $\Psi^{(i)}$ ($i=1,2,3$), we compute the asymptotic form, as $r\to\infty$, of the incoming part of the right-hand side of equation (6) and match the result against equation (8). We thus obtain

$${}^{+}K_{jm}^{(1)} = -\left(\frac{\pi}{2}(2J+1)\right)^{1/2}\frac{q}{|q|}\frac{\Gamma(a+\mathrm{i}\lambda)}{\Gamma(2a)}\frac{(2k)^{a}}{\mathrm{i}k}\exp[-\mathrm{i}(a-\mathrm{i}\lambda)\pi/2]\delta_{m+q+1/2,0} \tag{9a}$$

$${}^{+}K_{jm}^{(2)} = -\left(\frac{\pi}{2}(2J+1)\right)^{1/2}\frac{q}{|q|}\frac{\Gamma(a+1+\mathrm{i}\lambda)}{\Gamma(2a+2)}\frac{(2k)^{a+1}}{\mathrm{i}k}$$

$$\times\exp[-\mathrm{i}(a+1-\mathrm{i}\lambda)\pi/2]\delta_{m+q+1/2,0} \tag{9b}$$

$${}^{-}K_{jm}^{(1)} = {}^{+}K_{j,m-1}^{(1)} \qquad {}^{-}K_{jm}^{(2)} = -{}^{+}K_{j,m-1}^{(2)} \tag{9c}$$

$${}^{\pm}K_{m}^{(3)} = \pm 2\sqrt{\pi}(|q|\mp q)^{1/2}\Gamma(1+\mathrm{i}\lambda)\exp(-\pi\lambda/2)\delta_{m+q\pm1/2,0}. \tag{9d}$$

our scattering wavefunction is now completely determined. It is given by equation (6) with $\Psi_{jm}^{(i)}$ ($i=1,2,3$) given by equation (4) and the coefficients ${}^{\pm}K_{jm}^{i}$ ($i=1,2,3$) by equation (9).

4. Scattering amplitude

The outgoing part of the scattering wavefunction is defined by

$${}^{\pm}\Psi = {}^{\pm}\Psi_{\mathrm{inc}} + {}^{\pm}\Psi_{\mathrm{out}} \tag{10}$$

and the scattering amplitude $f^{\pm}$ is given by

$${}^{\pm}\Psi_{\mathrm{out}} \to \frac{\exp[\mathrm{i}(kr-\lambda\ln 2kr)]}{r}f^{\pm}(\theta) \tag{11}$$

as $r\to\infty$. Calculating the asymptotic form of ${}^{\pm}\Psi$ using (5) and comparing with (11) we obtain the scattering amplitude

$$f^{\pm}(\theta)=\frac{\mathrm{i}}{k}\frac{q}{|q|}\left(\frac{\pi}{2}\right)^{1/2}\left[\sum_{J=|q|+1/2}^{\infty}(2J+1)^{1/2}\,\mathrm{e}^{-\mathrm{i}a\pi}\left(\frac{\Gamma(a+\mathrm{i}\lambda)}{\Gamma(a-\mathrm{i}\lambda)}\,\xi^{(1)}_{jm}\mp\frac{\Gamma(a+1+\mathrm{i}\lambda)}{\Gamma(a+1-\mathrm{i}\lambda)}\,\xi^{(2)}_{jm}\right)\right.$$
$$\left.+\sqrt{2}(|q|\mp q)^{1/2}\frac{\Gamma(1+\mathrm{i}\lambda)}{\Gamma(1-\mathrm{i}\lambda)}\,\eta_m\right] \tag{12}$$

where $m=-q-\frac{1}{2}$ for f^{+} and $m=-q+\frac{1}{2}$ for f^{-}. We wish now to analyse $f^{\pm}$ in terms of amplitudes that describe scattering in which the outgoing fermion has definite helicity. Let $\zeta^{\pm}$ denote the normalised helicity eigenstates. Following the treatment in Kazama *et al* (1977), we can then find the connection between the helicity eigenstates $\zeta^{\pm}$ and the angular sections ξ^{i}_{jm} $(i=1,2)$ and η_m. We thus obtain for the case $m=-q-\frac{1}{2}$

$$\xi^{(1)}_{jm}-\xi^{(2)}_{jm}=\frac{\sqrt{2}\,qa}{|q|[(2J+1)(1-\cos\theta)]^{1/2}}$$
$$\times\left(\frac{1}{\sqrt{J}}\,Y_{J-1/2,m+1/2}-\frac{1}{\sqrt{J-1}}\,Y_{J+1/2,m+1/2}\right)\zeta^{+} \tag{13a}$$

$$\xi^{(1)}_{jm}+\xi^{(2)}_{jm}=-\frac{q\,\mathrm{e}^{-\mathrm{i}\phi}}{\sqrt{2}[(2J+1)(1+\cos\theta)]^{1/2}}$$
$$\times\left(\frac{2J+1+2q}{\sqrt{J}}\,Y_{J-1/2,m+1/2}+\frac{2J+1-2q}{\sqrt{J+1}}\,Y_{J+1/2,m+1/2}\right)\zeta^{-}. \tag{13b}$$

In the above, Y is the monopole spherical harmonics (Wu and Yang 1976) Y_{qJm} with the index q suppressed. Similarly, for the case $m=-q+\frac{1}{2}$, we find

$$\xi^{(1)}_{jm}-\xi^{(2)}_{jm}=\frac{q}{\sqrt{2}|q|}\left(\frac{1-\cos\theta}{2J+1}\right)^{1/2}\frac{\mathrm{e}^{\mathrm{i}\phi}}{\sin\theta}$$
$$\times\left(\frac{2J+1-2q}{\sqrt{J}}\,Y_{J-1/2,m-1/2}+\frac{2J+1+2q}{\sqrt{J+1}}\,Y_{J+1/2,m-1/2}\right)\zeta^{+} \tag{13c}$$

$$\xi^{(1)}_{jm}+\xi^{(2)}_{jm}=\frac{\sqrt{2}qa}{|q|\sin\theta}\left(\frac{1+\cos\theta}{2J+1}\right)^{1/2}\left(\frac{1}{\sqrt{J}}\,Y_{J-1/2,m-1/2}-\frac{1}{\sqrt{J+1}}\,Y_{J+1/2,m-1/2}\right)\zeta^{-}. \tag{13d}$$

As for η_m, we have the results

$$\eta_{-q-1/2}=-\left(\frac{2}{1+\cos\theta}\right)^{1/2}\left(\frac{|q|-q}{2|q|+1}\right)^{1/2}\mathrm{e}^{-\mathrm{i}\phi}\,Y_{|q|,-q}\zeta^{-} \tag{13e}$$

$$\eta_{-q+1/2}=-\sqrt{2}(1-\cos\theta)^{1/2}\frac{\mathrm{e}^{\mathrm{i}\phi}}{\sin\theta}\left(\frac{|q|+1}{2|q|+1}\right)^{1/2}Y_{|q|,-q}\zeta^{+}. \tag{13f}$$

Use of the above relations allows us to pick up the helicity-flip and helicity-nonflip amplitudes from (12). We thus obtain

$$f^{++}\equiv f^{+\to+}=\sqrt{\frac{\pi}{2}}\,\frac{\mathrm{i}}{2k}\sec\tfrac{1}{2}\Theta\,T_q \tag{14a}$$

$$f^{--}\equiv f^{-\to-}=f^{++} \tag{14b}$$

$$f^{+-}\equiv f^{+\to-}=-\sqrt{\frac{\pi}{2}}\,\frac{\mathrm{i}}{k}\,\mathrm{e}^{-\mathrm{i}\phi}\,\mathrm{cosec}\,\tfrac{1}{2}\Theta\left(\tfrac{1}{4}S_q-\sqrt{2}\,\frac{|q|-1}{(2|q|+1)^{1/2}}\,Y_{|q|,-q}\,\frac{\Gamma(1+\mathrm{i}\lambda)}{\Gamma(1-\mathrm{i}\lambda)}\right) \tag{14c}$$

$$f^{-+} \equiv f^{-\to +} = \sqrt{\frac{\pi}{2}}\,\frac{\mathrm{i}}{k}\,\mathrm{e}^{\mathrm{i}\phi}\operatorname{cosec}\tfrac{1}{2}\Theta\left(\tfrac{1}{4}\bar{S}_q - \sqrt{2}\,\frac{|q|+1}{(2|q|+1)^{1/2}}\,Y_{|q|,-q}\,\frac{\Gamma(1+\mathrm{i}\lambda)}{\Gamma(1-\mathrm{i}\lambda)}\right) \tag{14d}$$

where

$$T_q = \sum_{J=|q|+1/2}^{\infty} a\mathrm{e}^{-\mathrm{i}\pi a}\left(\frac{\Gamma(a+\mathrm{i}\lambda)}{\Gamma(a-\mathrm{i}\lambda)} + \frac{\Gamma(a+1+\mathrm{i}\lambda)}{\Gamma(a+1-\mathrm{i}\lambda)}\right)\left(\frac{1}{\sqrt{J}}\,Y_{J-1/2,-q} - \frac{1}{\sqrt{J+1}}\,Y_{J+1/2,-q}\right) \tag{15}$$

$$S_q = \sum_{J=|q|+1/2}^{\infty} \mathrm{e}^{-\mathrm{i}\pi a}\left(\frac{\Gamma(a+\mathrm{i}\lambda)}{\Gamma(a-\mathrm{i}\lambda)} - \frac{\Gamma(a+1+\mathrm{i}\lambda)}{\Gamma(a+1-\mathrm{i}\lambda)}\right) \times\left(\frac{2J+1+2q}{\sqrt{J}}\,Y_{J-1/2,-q} + \frac{2J+1-2q}{\sqrt{J+1}}\,Y_{J+1/2,-q}\right) \tag{16}$$

$$\bar{S}_q = \sum_{J=|q|+1/2}^{\infty} \mathrm{e}^{-\mathrm{i}\pi a}\left(\frac{\Gamma(a+\mathrm{i}\lambda)}{\Gamma(a-\mathrm{i}\lambda)} - \frac{\Gamma(a+1+\mathrm{i}\lambda)}{\Gamma(a+1-\mathrm{i}\lambda)}\right) \times\left(\frac{2J+1-2q}{\sqrt{J}}\,Y_{J-1/2,-q} + \frac{2J+1+2q}{\sqrt{J+1}}\,Y_{J+1/2,-q}\right) \tag{17}$$

and $\Theta = \pi - \theta$ is the scattering angle. From the property $Y^*_{qJm} = (-1)^{q+m}Y_{-q,J,-m}$ (Wu and Yang 1976) it is easy to see that *all* the monopole spherical harmonics that appear in equations (14)–(17) are even in q, and therefore

$$T_{-q} = T_q \qquad S_{-q} = \bar{S}_q \tag{18}$$

which leads to

$$f^{++}(-q) = f^{++}(q) \qquad f^{+-}(-q) = -\mathrm{e}^{-2\mathrm{i}\phi}f^{-+}(q). \tag{19}$$

In view of the above, it is thus sufficient to confine ones attention to the case $q \geqslant 0$. With this restriction on q, we shall now obtain the final form of our scattering amplitudes. We put in the expressions for the monopole spherical harmonics in terms of the Jacobi polynomials (Wu and Yang 1976) and thus obtain

$$\frac{1}{\sqrt{J}}\,Y_{J-1/2,-q} - \frac{1}{\sqrt{J+1}}\,Y_{J+1/2,-q} = \frac{2^{-q}}{\sqrt{2\pi}}(1+x)^q(p_n^{0,2q} - p_{n+1}^{0,2q})$$
$$= \frac{2^{-q}}{\sqrt{2\pi}}(1+x)^q(1-x)\,\frac{n+q+1}{n+1}\,p_n^{1,2q} \tag{20}$$

$$\frac{2J+1+2q}{\sqrt{J}}\,Y_{J-1/2,-q} + \frac{2j+1-2q}{\sqrt{J+1}}\,Y_{J+1/2,-q}$$
$$= \frac{2^{-q}}{\sqrt{2\pi}}\,2(1+x)^q[(n+2q+1)p_n^{0,2q} + (n+1)p_{n+1}^{0,2q}]$$
$$= \frac{2^{1-q}}{\sqrt{2\pi}}(1+x)^{q+1}(n+q+1)p_n^{0,2q+1} \tag{21}$$

$$\frac{2J+1-2q}{\sqrt{J}}\,Y_{J-1/2,-q} + \frac{2J+1+2q}{\sqrt{J+1}}\,Y_{J+1/2,-q}$$
$$= \frac{2^{1-q}}{\sqrt{2\pi}}(1+x)^q[(n+1)p_n^{0,2q} + (n+2q+1)p_{n+1}^{0,2q}]$$
$$= \frac{2^{1-q}}{\sqrt{2\pi}}(1+x)^q(2n+2q+2)p_{n+1}^{0,2q-1}. \tag{22}$$

In the above $x = \cos\theta = -\cos\Theta$ and $n = J - q - \frac{1}{2}$. In the second step of the equations (20)-(22), we used standard identities for the Jacobi polynomials (Erdélyi 1953). Finally, the above expressions are valid in the region a which is the sphere minus a strip around the south pole. In region b, which is the sphere minus a strip around the north pole, the right-hand sides of equations (20)-(22) have to be multiplied by the factor $\exp(-2iq\phi)$. Collecting our results (14)-(22) and introducing the notation

$$\Gamma(a+1+i\lambda) = |\Gamma(a+1+i\lambda)|\, e^{i\sigma_a} \qquad b = \frac{a-i\lambda}{a+i\lambda} \tag{23}$$

we now write down the final form of our scattering amplitudes:

$$f^{++} = \frac{i}{2k}\cos\tfrac{1}{2}\Theta(\sin\tfrac{1}{2}\Theta)^{2q} \sum_{n=0}^{\infty} a(1+b)\, e^{-i\pi a}\, e^{2i\sigma_a} \frac{n+q+1}{n+1} p_n^{1,2q}(\cos\theta) \tag{24a}$$

$$f^{--} = f^{++} \tag{24b}$$

$$f^{+-} = \frac{i}{2k} e^{-i\phi}(\sin\tfrac{1}{2}\Theta)^{2q+1} \sum_{n=0}^{\infty} (1-b)\, e^{-i\pi a}\, e^{2i\sigma_a}(n+q+1) p_n^{0,2q+1}(\cos\theta) \tag{24c}$$

$$f^{-+} = \frac{i}{2k} e^{i\phi}(\sin\tfrac{1}{2}\Theta)^{2q-1}\left(\sum_{n=0}^{\infty} (b-1)\, e^{-i\pi a}\, e^{2i\sigma_a}(n+q+1) p_{n+1}^{0,2q-1}(\cos\theta) - 2q\, e^{2i\sigma_0}\right). \tag{24d}$$

Note that in terms of the new summation variable, a is given by

$$a = [(n+1)(n+1+2q)]^{1/2}. \tag{25}$$

Moreover, the Jacobi polynomials can be expressed in terms of the scattering angle Θ by use of the identity

$$p_n^{\alpha,\beta}(-x) = (-1)^n p_n^{\beta,\alpha}(x). \tag{26}$$

We also note that the effect of dyon's electric charge is quite prominent in the helicity-flip amplitudes. We should now check two limiting cases.

In the limit of vanishing dyon electric charge ($Q=0$)$\lambda = 0$ and consequently $b = 1$. Now f^{+-} vanishes identically as does the first term in (24d) for f^{-+} and f^{++} undergoes corresponding simplification. The resulting expressions are identical with those obtained by Kazama *et al* (1977) for the monopole fermion scattering. Of course, the latter authors had treated the problem relativistically, by solving the Dirac equation, whereas our treatment is non-relativistic. But this distinction leads only to a kinematic difference in the relation that connects the momentum k with energy. Expressed in terms of the variable k, the scattering amplitudes are one and the same.

A second limiting case to study is the vanishing of the magnetic charge g of the dyon. Now $q = 0$ and $a = n+1$ and the relevent Jacobi polynomials are expressible in terms of Legendre polynomials as follows:

$$p_n^{1,0}(x) = \frac{1}{1-x}(p_n(x) - p_{n+1}(x)) \tag{27a}$$

$$p_n^{0,1}(x) = \frac{1}{1+x}(p_n(x) + p_{n+1}(x)) \tag{27b}$$

$$p_n^{-1,0}(x) = \tfrac{1}{2}(p_n(x) - p_{n-1}(x)). \tag{27c}$$

We see from (24) and (27) that

$$f^{-+} = -e^{2i\phi} f^{+-} \tag{28a}$$

$$f^{+-} = \frac{i}{4k} e^{-i\phi} \operatorname{cosec} \tfrac{1}{2}\Theta \sum_{n=0}^{\infty} (n+1)(b-1)\, e^{2i\sigma_{n+1}}(p_n(\cos\Theta) - p_{n+1}(\cos\Theta)) \tag{28b}$$

$$f^{++} = -\frac{i}{4k} \sec \tfrac{1}{2}\Theta \sum_{n=0}^{\infty} (n+1)(b+1)\, e^{2i\sigma_{n+1}}(p_n(\cos\Theta) + p_{n+1}(\cos\Theta)). \tag{28c}$$

It is convenient to analyse the above relations by introducing the following combinations of helicity amplitudes:

$$g^{++} = \cos\tfrac{1}{2}\Theta f^{++} - e^{i\phi}\sin\tfrac{1}{2}\Theta f^{+-} \tag{29a}$$

$$g^{+-} = e^{-i\phi}\sin\tfrac{1}{2}\Theta f^{++} + \cos\tfrac{1}{2}\Theta f^{+-} \tag{29b}$$

$$g^{-+} = \cos\tfrac{1}{2}\Theta f^{-+} - e^{i\phi}\sin\tfrac{1}{2}\Theta f^{--} \tag{29c}$$

$$g^{--} = e^{-i\phi}\sin\tfrac{1}{2}\Theta f^{-+} + \cos\tfrac{1}{2}\Theta f^{--}. \tag{29d}$$

Actually, g^{++}, g^{--} are the spin-non-flip and g^{+-}, g^{-+} the spin-flip amplitudes, as is not hard to verify. From (28) and (29) it follows directly that

$$g^{++} = g^{--} \qquad g^{-+} = -e^{2i\phi} g^{+-}. \tag{30}$$

Using the result

$$b\, e^{2i\sigma_{n+1}} = e^{2i\sigma_n} \tag{31}$$

we now put g^{+-} into the form

$$g^{+-} = \frac{i}{4k}\frac{e^{-i\phi}}{\sin\Theta} \sum_{n=0}^{\infty} (n+1)[e^{2i\sigma_n}(\cos\Theta p_n - p_{n+1}) + e^{2i\sigma_{n+1}}(\cos\Theta p_{n+1} - p_n)]. \tag{32}$$

By use of the identity

$$x p_n(x) = \frac{1}{2n+1}[(n+1)p_{n+1}(x) + n p_{n-1}(x)] \tag{33}$$

it is easily proved that

$$g^{+-} = 0. \tag{34}$$

Again use of (31) leads to the following

$$g^{++} = -\frac{i}{2k}\sum_{n=0}^{\infty} (n+1)(e^{2i\sigma_{n+1}} p_{n+1} + e^{2i\sigma_n} p_n) = -\frac{i}{2k}\sum_{n=0}^{\infty} (2n+1)\, e^{2i\sigma_n} p_n(\cos\Theta). \tag{35}$$

The above series is well known. Specifically

$$g^{++} = g^{--} = -\frac{\lambda}{k} e^{2i\sigma_0} 2^{i\lambda} \frac{(1-\cos\Theta)^{-i\lambda}}{1-\cos\Theta} \tag{36}$$

which is recognised to be the correct Coulomb scattering amplitude that leads to the Rutherford formula for the differential cross section.

3. Electron capture by dyons

Let us consider the process in which an incoming electron is captured into a bound orbit by a positively charged dyon, with the emission of a photon. We propose to calculate the cross section for this process, in the dipole approximation, for the case where the capture takes place into the ground state of the dyon–electron bound system. In the dipole approximation the capture cross section is

$$\sigma = \tfrac{4}{3}\alpha \frac{M}{k} \omega^3 |\langle f|\boldsymbol{r}|i\rangle|^2 \tag{37}$$

where $\omega = E_i - E_f = E_i + |E_f|$ is the photon energy, α the fine structure constant, k the incoming electron momentum. Introducing the 'Bohr' radius $a_0 = 1/MeQ$, equation (37) can be written as

$$\sigma = \frac{\alpha}{6} \frac{|\lambda|}{M^2 a_0^5} \left(1 + \frac{1}{\lambda^2}\right)^3 |\langle f|\boldsymbol{r}|i\rangle|^2 \tag{38}$$

where λ has been defined in (4). The initial state $|i\rangle$ that appears above is given by equations (6) and (9) of section 2. The final ground state of the sound system has already been calculated by us (Bose 1986). Thus we have everything needed for the evaluation of the dipole matrix element. Finally, we have to average over the two initial-state configurations for initially unpolarised electrons.

We consider the case $q = \pm\frac{1}{2}$. The final state is

$$\Psi_f = \sqrt{MeQ}\, 2k' \, e^{-k'r} \eta_0 \tag{39}$$

where $k' = (-2ME_f)^{1/2}$. The dipole matrix element factorises into radial and angular parts. Clearly, only the $j = 1$ terms in the initial state survive the angular integration. The corresponding angular matrix elements are

$$\langle \eta_0 | Y_{1a} | \xi^{(1)}_{1m} \rangle = -\delta_{m+a,0} \frac{(-1)^a}{\sqrt{8\pi}} \qquad |q| = \tfrac{1}{2} \tag{40a}$$

$$\langle \eta_0 | Y_{1a} | \xi^{(2)}_{1m} \rangle = \pm\delta_{m+a,0} \frac{(-1)^a}{\sqrt{8\pi}} \qquad q = \pm\tfrac{1}{2}. \tag{40b}$$

The radial integrals appearing, respectively, in the matrix elements $\langle f|\boldsymbol{r}|\psi^{(1)}_{1m}\rangle$ and $\langle f|\boldsymbol{r}|\psi^{(2)}_{1m}\rangle$ are given by

$$\int_0^\infty \mathrm{d}r\, r^{2+\sqrt{2}} \exp[-(k'+\mathrm{i}k)r]\, {}_1F_1(\sqrt{2} - \mathrm{i}\lambda, 2\sqrt{2}, 2\mathrm{i}kr) \tag{41a}$$

$$\int_0^\infty \mathrm{d}r\, r^{3+\sqrt{2}} \exp[-(k'+\mathrm{i}k)r]\, {}_1F_1(\sqrt{2} + 1 - \mathrm{i}\lambda, 2\sqrt{2}+2, 2\mathrm{i}kr). \tag{41b}$$

The integrals can be evaluated closely provided k is suitably small, namely, $k' > \sqrt{3}k$ by use of the formula (Gradshteyn and Ryzhik 1965)

$$\int_0^\infty e^{-st} t^{b-1}\, {}_1F_1(a, c, kt)\, \mathrm{d}t = \Gamma(b) s^{-b} F(a, b, c, ks^{-1}) \tag{42}$$

where F is the hypergeometric function and the result is valid provided the condition

$|s|>|k|$ is satisfied. Collecting our results (40)-(42) together, we have

$$\langle\Psi_{\mathrm{ground}}|r^a|\Psi^{(1)}_{1m}\rangle=\left(\frac{2MeQ}{3}\right)^{1/2}(-1)^{a+1}\frac{k'}{(k'+\mathrm{i}k)^{3+\sqrt{2}}}\Gamma(3+\sqrt{2})F_{(1)}\delta_{m,-a} \tag{43a}$$

$$\langle\Psi_{\mathrm{ground}}|r^a|\Psi^{(2)}_{1m}\rangle=\pm\left(\frac{2MeQ}{3}\right)^{1/2}(-1)^{a}\frac{k'}{(k'+\mathrm{i}k)^{4+\sqrt{2}}}\Gamma(4+\sqrt{2})F_{(2)}\delta_{m,-a} \tag{43b}$$

where the abbreviations

$$F_{(1)}=F\left(\sqrt{2}-\mathrm{i}\lambda,3+\sqrt{2},2\sqrt{2},\frac{2\mathrm{i}k}{k'+\mathrm{i}k}\right) \tag{44a}$$

$$F_{(2)}=\left(\sqrt{2}+1-\mathrm{i}\lambda,4+\sqrt{2},2\sqrt{2}+2,\frac{2\mathrm{i}k}{k'+\mathrm{i}k}\right) \tag{44b}$$

have been used, and the $\pm$ sign in (43*b*) (corresponds to $q=\pm\frac{1}{2}$. It is now straightforward to compute the cross section. It is instructive to consider the limiting case of a very slowly moving incoming electron, $k\to 0$. It is intuitively obvious that this is the most favourable setup for capture. In this limit $\lambda\to\infty$ and the hypergeometric functions go over to the confluent hypergeometric functions, $F(a, b, c, x)\to {}_1F_1(b, c, ax)$ as $|a|\to\infty$. The resulting expressions are then easy to evaluate numerically. In this manner, we obtain for the capture cross section for initially unpolarised electrons the limiting result

$$\sigma=5.7\alpha\left(\frac{eQ}{k}\right)^2. \tag{45}$$

For the sake of comparison, the corresponding capture cross section for a proton (leading to the formation of a hydrogen atom) is

$$\sigma_{\mathrm{H}}=15.4\alpha\left(\frac{e^2}{k}\right)^2. \tag{46}$$

Thus the ratio of our cross section to that for hydrogen formation, compared at the same initial electron momentum, is 0.4 $(Q/e)^2$.

In conclusion, we have studied two aspects of the non-relativistic dyon-electron system. First, the scattering problem has been solved. The resulting scattering amplitudes have been shown to possess correct behaviour under two limiting cases. These are the limits where the electric charge of the dyon is set equal to zero and where the magnetic pole strength of the dyon is set equal to zero. The problem of electron capture by a positively charged dyon leading to the formation of a dyon-electron bound state has also also been solved for the special case where the bound system is in its ground state. In a subsequent publication, the capture cross section for excited states of the bound system as well as a computation of the recombination coefficient will be attempted.

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